The Use of Binary Numbers, Subpiles, and Nimbers
in the Optimal Strategy for Nim

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ABSTRACT

This paper describes the straight and common (misère) forms of basic Nim, how binary numbers can be used in three ways to calculate winning positions to execute an optimal strategy for Nim play, and variant end game play for common Nim. In addition, this paper provides a proof of concept for expanding Nimber tables beyond two dimensions for use in play with four or more nim heaps.

INTRODUCTION

Nim is a zero-sum game played by two people with a number of "tokens"[1] arranged in two or more rows (known as "nim heaps"[2]). Each player takes a turn removing one or more tokens from a single nim heap. Nim may be played in two forms (Beasley 1990)—(1) "common" Nim (a misère form where the loser is forced to take the last token) and (2) "straight" Nim, where the winner takes the last token. The different forms require a slight variation in end-game play--for example, in a simplified position, with one token in the first nim heap and two tokens in the second heap (a 1-2 position), the player taking the turn will win in the common Nim form by removing both tokens in the heap of two, forcing his opponent to take the remaining token in the other heap. However, in the straight Nim form, he would instead remove one of the two tokens in the heap of two, forcing his opponent to take the remaining token from one of the two remaining heaps, leaving the final token for the last play. Each of the optimal strategies discussed will only work until an end game position is reached—common Nim requires an end-game change in strategy discussed below.

In addition to the two forms discussed above, there are many Nim variants or games, some of which involve different rules of play and therefore different strategies. These variants will not be discussed in this paper.

BASIC NIM STRATEGY

Nim strategies are used to determine whether a given position is balanced or unbalanced, and what moves will balance the position for the opponent's turn. A player who is offered a balanced position is at a disadvantage; therefore, the objective in each turn is to offer your opponent a balanced position, which they have no choice but to unbalance. A balanced position[3], when modified by a player's turn, will always be unbalanced (Fuchs 1971). An unbalanced position, when modified by a player's turn, may be balanced or remain unbalanced. The flow of a Nim game, in balanced and unbalanced positions, is represented in figure 1.

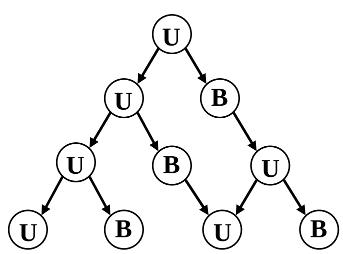

figure 1. Nim flow tree

[1] *The term "token" is used herein as a generic term to refer to whatever item is used in play (e.g., cards, coins, matchsticks, marbles, etc.).*

[2] *The term "nim heap" is used herein as a generic term to refer to a group of tokens upon which a player may act. A nim heap may be a row, column, pile, etc., depending on the variant of play.*

[3] *A balanced position is variously referred to in Nim literature as an "even position," an "unsafe position," or a "losing position." An unbalanced position is also referred to in Nim literature as an "uneven position," an "odd position," a "safe position," or a "winning position."*

However, if the starting position is balanced, or your opponent takes first and offers you a balanced position, you must count on your opponent to make an error (and leave you an unbalanced position) for a chance to employ a Nim strategy for a win. (In Alain Resnais' 1961 film, *L'année dernière à Marienbad* (*Last Year at Marienbad*), one of the characters introduces a Nim variant henceforth referred to as "Marienbad" with a 1-3-5-7 formation using playing cards and matchsticks as tokens. Because the starting position is balanced, the character who introduces the game graciously allows his opponent to take the first turn.)

OPTIMAL STRATEGIES FOR NIM

Binary Numbers. Binary numbers are used to evaluate whether a particular position is balanced or unbalanced, and if the position is unbalanced, what move can be made to balance the position for the opponent's play (Spencer 1968, Kraitchik 1942, Fuchs 1971). To employ binary numbers in an optimal strategy, the first step is to render the number of tokens in each nim heap as a binary number. For example, if the position is 3-5-7 (as in figure 2), write the number of tokens in each heap in binary. (All binary numbers should have the same number of digits; add 0s to the left of smaller numbers, if necessary, as in the first row of the example.)

Next, use an iterative similar/dissimilar (S/D) operation, where each heap is compared to the next (see figure 2). For example, in comparing the first two heaps, both contain a "1" in the first (read from the right) column. These similar digits are represented as a 0. In the second and third columns, one heap contains a "0," while the other contains a "1." These dissimilar digits are represented as a 1. Therefore, the comparison of the first two heaps, using the S/D operation, yields a sum of 110. Comparing the third heap with the sum of the first two heaps, in the first column, there is a "1" and a "0," which are dissimilar, and in the second and third columns, both numbers are "1," which are similar. Therefore, the nim sum, as calculated using the S/D operation, is 001, which tells us that the total position is unbalanced, and that one token must be removed from the first column to balance the position.

XXX	011
XXXXX	<u>101</u>
S/D composite of first two heaps:	**110**
XXXXXXX	<u>111</u>
S/D composite of entire position:	**001**

figure 2

A Nim-playing machine can be built that uses the S/D operation on binary numbers to calculate winning positions (Fuchs 1971). However, these operations can be difficult to visualize for human play.

Subpiles. The subpile strategy leverages the binary method to create a visual tool for quickly breaking down nim positions. Each nim heap is divided into subpiles, with each subpile containing an exact power of two. Thus, a heap with nine tokens is broken into a subpile with eight tokens (2^3) and a subpile with one token (2^0), a heap with six tokens is broken into a subpile with four tokens (2^2) and a subpile with two tokens (2^1), etc. In each case, a heap is broken into the minimum possible number of subpiles based on the power of two (that is, eight tokens must become one subpile of eight, not two subpiles of four). These correspond to the columns in the binary calculation.

To evaluate a position, tabulate the number of each size subpile—if there is an even number of each size subpile (for example, two subpiles of eight, two subpiles of two, and four subpiles of one), the position is balanced. If there is an odd number of any size subpile (for example, two subpiles of eight, one subpile of four, and three subpiles of two), the position is unbalanced.

For example, if the starting position consists of three heaps, with three, five, and seven tokens, respectively (a 3-5-7 position), as in figure 3 below, each heap is broken into subpiles based on powers of two, and the number of each size subpile is tabulated to evaluate the position (which, in this case, is unbalanced).

XXX	XX X	2 (2^1), 1 (2^0)	2^0 - 3
XXXXX	XXXX X	4 (2^2), 1 (2^0)	2^1 - 2
XXXXXXX	XXXX XX X	4 (2^2), 2 (2^1), 1 (2^0)	2^2 - 2

figure 3

To balance the position, the player must remove one occurrence of 2^0. This single token subpile is taken (from any pile where it occurs) to balance the position (see figure 4 below).

XXX	XX X	2 (2^1), 1 (2^0)	2^0 - **2**
XXXXX	XXXX ~~X~~	4 (2^2), ~~**1 (2^0)**~~	2^1 - 2
XXXXXXX	XXXX XX X	4 (2^2), 2 (2^1), 1 (2^0)	2^2 - 2

figure 4

Although the Subpile method is simply a different perspective of the binary method, it is easier to use for mental position calculation.

Nimbers. The binary method can also be simplified by using the binary position calculations to construct a nimber reference table, such as the one in figure 5. Nimbers make use of an algebraic equation, *a+*b+*n=*0, where *0 represents a balanced position, *a and *b represent the nimbers of tokens in two of the heaps, and *n represents the nimber of tokens in the third heap. In any configuration with three heaps, the intersection of the number of tokens in one heap with the number of tokens in a second heap is the number of tokens to which the third heap must be reduced to create a balanced position (Berlekamp et al 1982). For example, in a 3-5-7 configuration, the intersection of 3 and 5 is 6. Therefore, if the heap of 7 is reduced to a heap of 6, the configuration will be balanced. In addition, the intersection of 3 and 7 is 4, and the intersection of 5 and 7 is 2. Therefore, according to the nimber addition table, removing one token from any heap will balance a 3-5-7 configuration. If the intersection of the first and second heaps is higher than the current third heap, look up the intersection of the first and third or second and third heaps. For example, if the configuration is 3-4-5, the intersection of 3 and 4 yields 7. Tokens cannot be added to a heap; therefore, two different numbers must be checked—for example, 4 and 5. The intersection of 4 and 5 is 1, so to balance a 3-4-5 configuration, two tokens should be removed from the first heap.

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

figure 5

The nimber table can be expanded into additional dimensions to calculate balanced positions for four or more nim heaps (figure 6). Nimbers need not be used when the field of play is reduced to two heaps because the only balanced position is (x,x), where both heaps have the same number of tokens.

NIM END GAME

For common Nim, the unbalanced to balanced position strategy may need to be modified in the end game. An end game occurs when a balancing move would reduce the field of play to an even number of one-token heaps. For example, figure 7 shows a situation in which a player must recognize that a win can be achieved in common Nim by removing all three counters in the first heap, rather than balancing the position by removing only two of the counters from the first heap, as one would to win in straight Nim. If a player were to reduce the field to two one-token heaps while playing common Nim, his opponent would win by removing one of the two heaps, leaving the lone token in the remaining heap.

REFERENCES

- Ball, W. W. R., & In Coxeter, H. S. M. (1947). *Mathematical recreations & essays, rev. by H. S. M. Coxeter.* New York: Macmillan Co.
- Beasley, J. D. (1990). *The mathematics of games*. Oxford: Oxford University Press.
- Berlekamp, E. R., Conway, J. H., & Guy, R. K. (1982). *Winning ways, for your mathematical plays*. London: Academic Press.
- Eigen, M., & Winkler, R. (1981). *Laws of the game: How the principles of nature govern chance*. New York: Knopf.

Nim Heap X Contains 0 (3 Heaps)

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

Nim Heap X Contains 1

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	2	3	4	5	6	7
2	2	1	0	7	6	5	4
3	3	0	1	6	7	4	5
4	4	7	6	1	0	3	2
5	5	6	7	0	1	2	3
6	6	5	4	3	2	1	0
7	7	4	5	2	3	6	1

Nim Heap X Contains 2

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	2	1	0	7	6	5	4
2	1	2	3	4	5	6	7
3	0	3	2	5	4	7	6
4	7	4	5	2	3	0	1
5	6	5	4	3	2	1	0
6	5	6	7	0	1	2	3
7	4	7	6	1	0	3	2

Nim Heap X Contains 3

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	3	0	1	6	7	4	5
2	0	3	2	5	4	7	6
3	1	2	3	4	5	6	7
4	6	5	4	3	2	1	0
5	7	4	5	2	3	0	1
6	4	7	6	1	0	3	2
7	5	6	7	0	1	2	3

Nim Heap X Contains 4

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	4	7	6	1	0	3	2
2	7	4	5	2	3	0	1
3	6	5	4	3	2	1	0
4	1	2	3	4	5	6	7
5	0	3	2	5	4	7	3
6	3	0	1	6	7	4	5
7	2	1	0	7	3	5	4

Nim Heap X Contains 5

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	5	6	7	0	1	2	3
2	6	5	4	3	2	1	0
3	7	4	5	2	3	0	1
4	0	3	2	5	4	7	6
5	1	2	3	4	5	6	7
6	2	1	0	7	6	5	4
7	3	0	1	6	7	4	5

Nim Heap X Contains 6

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	6	5	4	3	2	1	0
2	5	6	7	0	1	2	3
3	4	7	6	1	0	3	2
4	3	0	1	6	7	4	5
5	2	1	0	7	6	5	4
6	1	2	3	4	5	6	7
7	0	3	2	5	4	7	6

Nim Heap X Contains 7

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	7	4	5	2	3	0	1
2	4	7	6	1	0	3	2
3	5	6	7	0	1	2	3
4	2	1	0	7	6	5	4
5	3	0	1	6	7	4	5
6	0	3	2	5	4	7	6
7	1	2	3	4	5	6	7

figure 6

XXX
X

~~XXX~~ X (Common)	X~~XX~~ X (Straight)

figure 7

- Fuchs, W. R. (1971). *Computers: information theory and cybernetics*. London: Hart-Davis.
- Hardy, G. H., & Wright, E. M. (1960). *An introduction to the theory of numbers*. Oxford: Clarendon Press.
- Kasner, E., & Newman, J. R. (1940). *Mathematics and the imagination*. New York: Simon and Schuster.
- Kraitchik, M. (1942). *Mathematical recreations*. New York: Dover Publications.
- Spencer, D. D. (1968). *Game playing with computers*. New York,N.Y: Spartan books.
- Bouton, C. L. (January 01, 1902). Nim, A Game with a Complete Mathematical Theory. *Annals of Mathematics, 3,* 4, 35-39.
- Moore, E. H. (April 01, 1910). A Generalization of the Game Called Nim. *Annals of Mathematics, 11,* 3, 93-94.